{ B.Sc Part I (Physics (Hons)) }
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Q (a) State and prove Kepler's laws of planetary motion using the concept of reduced mass

(b) Show that the areal velocity of planet around the sun is constant

(c) Show that the square of the time period of revolution of a planet is proportional to the cube of semi-major axis of orbit

Ans (a) Kepler's laws of planetary Motion:—

Kepler's three laws of ~~plant~~ planetary motions are :-

① Each planet moves in an ellipse with sun at its focus.

② The radius vector i.e the line joining the sun to the given planet sweeps out equal area in equal interval of time.

③ The square of the period of revolution of the planet about the Sun divided by the cube of the major axis of orbit is constant

First law :— The eccentricity of the orbit of a particle moving under attractive inverse square law forces is given by

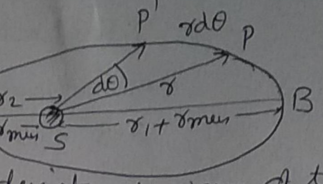
$$e^2 = 1 + \frac{2EJ^2}{mc^2}$$

where J is the angular momentum of the particle of reduced mass 'm'. E' be the total energy, c = constant which is in case of gravitational force = GMm, M being the mass of the body about which the particle moves and G be the gravitational Constant

$$\therefore E = -\frac{mc^2}{2J^2}(1-e^2)$$

In case of a planet revolving round the Sun, if e<1, the total energy of the system is (−ve), i.e the planet remain bound to the attracting centre, the Sun. As the planet is bound to the Sun and cannot escape from it. It moves around the Sun in a closed elliptical path. The establishes Kepler's I^st law of planetary motion.

In the case of the motion of a planet round the Sun, The Sun is at one focus which



is taken at the centre of the co-ordinate system of the ellipse. Thus each planet moves in an ellipse with Sun at its focus. The point A where the planet is closest to the Sun is called perihelion and the point B where it is farthest from the Sun is called apehelion.

(b) Second law :→ Suppose a planet P is moving in an elliptic orbit, it moves from P to P' in a small time dt, the area swept out by the radius vector is the area of the figure SPP'. If 'dt' is infinitesimally small PP' is a straight line $= r d\theta$ and SPP' is a ~~tra~~ triangle.

The area of the triangle SPP' $= dA = \frac{1}{2} r \cdot r d\theta = \frac{1}{2} r^2 d\theta$

This is the area swept $= \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$

The angular momentum $J = m r^2 \dot{\theta}$ and is a constant under a central force

$\therefore r^2 \dot{\theta} = \frac{J}{m}$ = Constant under gravitational force or

$\frac{1}{2} r^2 \dot{\theta} = \frac{J}{2m}$ = a constant

Hence $\frac{dA}{dt} = \frac{J}{2m}$ = a constant which verifies the 2nd law that the radius vector joining the sun to the planet sweeps out equal areas in equal intervals of time. In other words, the areal velocity of a planet around the Sun is constant

(c) Third law :-

As we proved above $\frac{dA}{dt} = \frac{J}{2m}$

$\therefore dA = \frac{J}{2m} dt$

$$\int_0^A dA = \int_0^T \frac{J}{2m} dt$$

$\therefore A = \frac{JT}{2m}$ where A = area of ellipse

T = Time period of one full revolution.

Now the area of ellipse $= \pi ab$ ∴ a = semi major axis, b = semi minor axis

$$\therefore \frac{JT}{2m} = \pi ab \quad \therefore T = \frac{2\pi m a b}{J}$$

Now $b = a\sqrt{1-e^2}$

$$\therefore T = \frac{2\pi m a^2\sqrt{1-e^2}}{J} \Rightarrow T^2 = \frac{4\pi^2 m^2 a^4 (1-e^2)}{J^2} \quad \text{---- (1)}$$

As the origin is taken as the focus S.

$$2a = r_{max} + r_{min}$$

Now, $\frac{1}{r} = \frac{mc}{J^2}\left(1 + \frac{AJ^2}{mc}\cos\theta\right) = \frac{mc}{J^2}(1 + e\cos\theta)$

$$\therefore \left(\frac{1}{r}\right)_{max} = \frac{mc}{J^2}(1+e)$$

$$\therefore r_{min} = \frac{J^2}{mc}\frac{1}{1+e}$$

Similarly $r_{max} = \frac{J^2}{mc}\frac{1}{1-e}$

Hence $2a = \frac{J^2}{mc}\left[\frac{1}{1+e} + \frac{1}{1-e}\right]$

$$\therefore a = \frac{J^2}{2mc}\left[\frac{1}{1+e} + \frac{1}{1-e}\right] = \frac{J^2}{2mc}\left[\frac{2}{1-e^2}\right] = \frac{J^2}{mc(1-e^2)}$$

Now from eqn (1)

$$T^2 = \frac{4\pi^2 m^2 a^4 (1-e^2)}{J^2} = \frac{4\pi^2 m^2 a^3}{J^2} \cdot \frac{J^2(1-e^2)}{mc(1-e^2)} = \frac{4\pi^2 m a^3}{c}$$

Hence $\frac{T^2}{a^3} = \frac{4\pi^2 m}{c}$ = a constant

Thus the square of the period of revolution of the planet about the Sun divided by the cube of major axis of the orbit is a constant which is Kepler's 3rd law.

or

The square of time period of revolution of a planet is proportional to the cube of semi major axis of the orbit.

$$T^2 \propto a^3$$